Problem of the Week

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Contents

Chapter 1

Satifying

1.1 Infinite Square Roots

Find the value of:

$$\sqrt{1-\sqrt{\frac{17}{16}-\sqrt{1-\sqrt{\frac{17}{16}-\sqrt{1-\dots}}}}} \tag{1.1}$$

Assume, when taking the limit, that deep within the nested square roots we choose to begin with a "1"

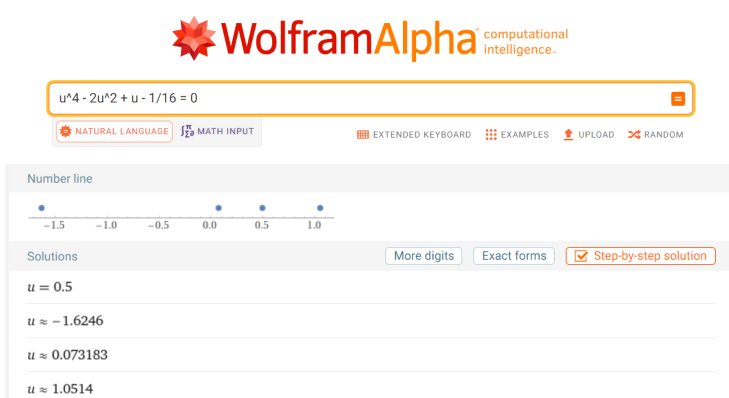
1.1.1 Solution

We can do a substitution for U

$$U = \sqrt{1-\sqrt{\frac{17}{16}-\sqrt{1-\sqrt{\frac{17}{16}-\sqrt{1-\dots}}}}}$$

$$U = \sqrt{1-\sqrt{\frac{17}{16}-U}} \qquad \text{Using definition above}$$

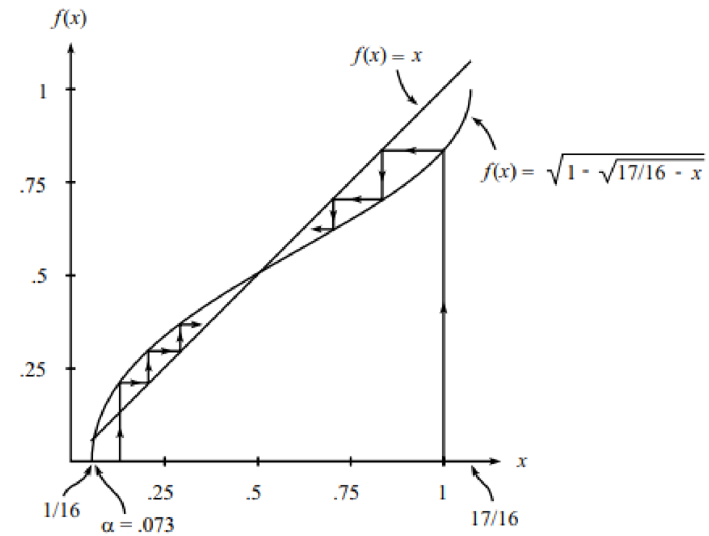
$$0 = U^4 - 2U^2 + U - \frac{1}{16} \qquad \text{Algebraic simplification}$$



So the solution must be one of the four options. Off the bat, we can eliminate two of these answers. $U \approx 1.05$ doesn't make sense since it's the form $\sqrt{1-(\dots)}$ and the fact the $(\dots)$ is positive implies that $U < 1$. Secondly, $U \approx -1.62$ doesn't make sense as it would imply that $\sqrt{(\dots)}$ results in a negative number. (These are solution to the expression in polynomial form, but in our original "infinite square roots" expression it doesn't make sense)

So we've eliminated two options and we have $U = 1/2$ and $U \approx 0.073$. The solution 1/2 is more stable. We can see why by first defining a new function:

$$f(x) = \sqrt{1 - \sqrt{\frac{17}{16} - x}}$$



The problem stated that deeply nested in the square roots, we'd begin with a one. For $0.73... < x_0 \leq 17/16$, if I keep applying f (for example, $f(f(f(f(x_0))))$ etc), the result would converge to 1/2. So for our problem, repeated application of f on $x_0 = 1$ will result in $\approx 1/2$. As we keep applying the function to one, we get 1/2. Hence, $x = 1/2$.

$$\sqrt{1 - \sqrt{\frac{17}{16} - \sqrt{1 - \sqrt{\frac{17}{16} - \sqrt{1 - ...}}}}} = \frac{1}{2} \tag{1.2}$$

Reference: David Morin, Co-Director of Undergraduate Studies, Senior Lecturer on Physics (Harvard). https://www.physics.harvard.edu/files/sol78.pdf

Chapter 2

Curiosity

2.1 Deriving the Normal Distribution

The normal distribution probability density function is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{2.1}$$

How is this derived? And why on Earth is π in this function?

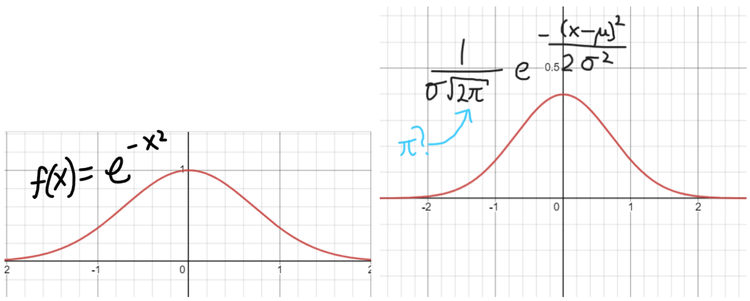


Figure 2.1: We'll try to turn the function on the left into the function on the right

Let's first take a step back and examine $f(x) = e^{-x^2}$ which has the bell shaped curve that we want. Somehow, this particular shape tends to describe various phenomena in nature, so we use it. Let's bend this function to satisfy some properties.

1. **Satisfying the inflection property.** The normal distribution has the special property that the inflection points occur at $x = \pm\sigma$ when $\mu = 0$. Let $f(x) = e^{-x^2/k}$. Find out what k needs to be in order for the inflection points to occur at $\pm\sigma$.

2. **Solve the Gaussian Integral.** The Gaussian Integral is a variation of our probability density function, so solving this easier Gaussian Integral gives a method to solve the PDF. $I = \int_{-\infty}^{\infty} e^{-x^2} dx$. What is I? (Hint: note that this integral $\int e^{-x^2} dx$ famously has no elementary function as a solution. However, $I \cdot I$ is a much easier integral to solve for in terms of x and y and converting to polar.)

3. **Satisfying summing to one property.** $\int_{-\infty}^{\infty} f(x)dx = 1$ property as a property of a probability distribution. Now combine parts 1) and 2) by solving for $I' = \int_{-\infty}^{\infty} e^{-x^2/k} dx$ using the same steps in 2) except substitute what you got in 1) for k. By finding I', we can satisfy the summing to one property.

2.1.1 Solution

Inflections Property

We want the inflection points to occur at $x = \pm\sigma$ assuming $\mu = 0$.

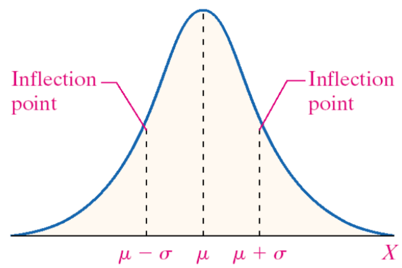


Figure 2.2: Let's assume $\mu = 0$

Inflection points occur when $f''(x) = 0$. Let's find $f''(x)$ first.

$$\begin{aligned} f(x) &= e^{-x^2/k} \\ f'(x) &= -\frac{2x}{k}e^{-x^2/k} \\ f''(x) &= -\frac{2}{k}e^{-x^2/k} + \frac{4x^2}{k^2}e^{-x^2/k} \end{aligned}$$

Now we can set $f''(x) = 0$:

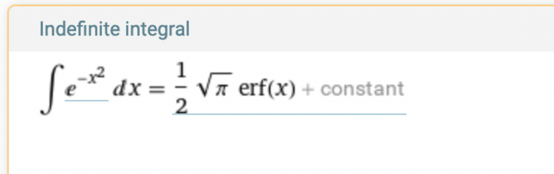
$$\begin{aligned} 0 &= -\frac{2}{k}e^{-x^2/k} + \frac{4x^2}{k^2}e^{-x^2/k} \\ \frac{2}{k}e^{-x^2/k} &= \frac{4x^2}{k^2}e^{-x^2/k} \\ 1 &= \frac{2x^2}{k} \\ \sqrt{\frac{k}{2}} &= x && \text{at the inflection point} \\ \sqrt{\frac{k}{2}} &= \pm\sigma && \text{inflection points occur at } x = \pm\sigma \\ k &= 2\sigma^2 \end{aligned}$$

So now we have that $f(x) = e^{-x^2/(2\sigma^2)}$, where inflections for this occur at $x = \pm\sigma$.

Solving the Gaussian Integral

$$I = \int_{-\infty}^{\infty} e^{-r^2} dr \tag{2.2}$$

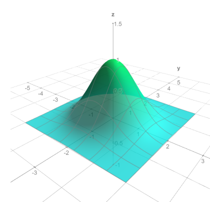
This integral $\int e^{-x^2}$ famously has a non-elementary solution. If you put it directly into WolframAlpha, you get an error function.



However $\int_{-\infty}^{\infty} e^{-r^2} dr$ contains a neat solution. Instead let's solve for $I \cdot I$:

$$\begin{aligned} I \cdot I &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \end{aligned}$$

This is particularly helpful because we can interpret this as a rotation around the z axis where $x^2 + y^2 = r^2$ and r is the distance from the origin.



$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy &= \int_{0}^{\infty} \int_{0}^{2\pi} e^{-r^2} r d\theta dr \\ &= \int_{0}^{\infty} 2\pi e^{-r^2} r dr \\ &= \pi \int_{0}^{\infty} e^{-u} du && u = r^2, du = 2r dr \\ &= -\pi \left(e^{-u}\right)_{0}^{\infty} \\ &= -\pi \left(0 - 1\right) \\ &= \pi \end{aligned}$$

This means that

$$\begin{aligned} I \cdot I &= \pi \\ I &= \sqrt{\pi} \end{aligned}$$

Sums to one property

Let's redo the Gaussian integral using $f(u) = e^{-u^2/2\sigma^2}$

$$\begin{aligned}
I' \cdot I' &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2\sigma^2} dxdy \\
&= \int_{0}^{\infty} \int_{0}^{2\pi} e^{-r^2/2\sigma^2} r d\theta dr \\
&= \int_{0}^{\infty} 2\pi e^{-r^2/2\sigma^2} r dr \\
&= 2\pi\sigma^2 \int_{0}^{\infty} e^{-r^2/2\sigma^2} \frac{r}{\sigma^2} dr \qquad\qquad u = r^2/2\sigma^2, du = \frac{r}{\sigma^2} dr \\
&= 2\pi\sigma^2 \int_{0}^{\infty} e^{-u} du \\
&= -2\pi\sigma^2 \left(e^{-u}\right)_0^{\infty} \\
&= -2\pi\sigma^2 (0-1) \\
&= 2\pi\sigma^2
\end{aligned}$$

Hence:

$$\begin{aligned}
I' \cdot I' &= 2\pi\sigma^2 \\
I' &= \sigma\sqrt{2\pi}
\end{aligned}$$

But we want the integral to sum to one. Multiplying by $\frac{1}{\sigma\sqrt{2\pi}}$ solves this problem. $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$ solves this problem.

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)} dx = 1 \tag{2.3}$$

Final Expression

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)} \tag{2.4}$$

Adding μ translates the bell curve among the x axis.

References

Thank you Vedant Jhawar (UC Berkeley Math and CS) for proofreading. The solution was mostly inspired by a vcubingx / 3b1b video. https://youtu.be/d$_q vLDhkg$00. Some parts were derived from first principles.

Chapter 3

Novelty

3.1 Does $e^\pi < \pi^e$?

Simple question. Without a calculator, show whether

$$e^\pi < \pi^e? \tag{3.1}$$

3.1.1 Solution

One simple solution

$$\begin{array}{rcl} x^y & ? & y^x \\ \ln(x^y) & ? & \ln(y^x) \\ y\ln(x) & ? & x\ln(y) \\ \frac{\ln(x)}{x} & ? & \frac{\ln(y)}{y} \end{array}$$

For the function $f(x) = \ln(x)/x$, we can find that there exists a global maximum at $x = e$.

$$\begin{aligned} f(x) &= \frac{\ln(x)}{x} \\ \frac{df}{dx} &= \frac{1-\ln(x)}{x^2} \\ \frac{df}{dx}|_{x=e} &= 0 \\ \frac{d^2f}{dx^2} &= \frac{-3+2\ln(x)}{x^3} \\ \frac{d^2f}{dx^2}|_{x=e} &< 0 \end{aligned}$$

Because of the global maximum,

$$\frac{\ln(e)}{e} \quad > \quad \frac{\ln(\pi)}{\pi} \tag{3.2}$$

Hence the solution is:

$$e^\pi > \pi^e \tag{3.3}$$

Another Solution

$$
\begin{aligned}
x^y \quad &? \quad y^x \\
\frac{\ln(x)}{x} \quad &? \quad \frac{\ln(y)}{y} && \text{Look above for steps} \\
\frac{\ln(x)}{x} - \frac{\ln(y)}{y} \quad &? \quad 0 \\
\int_\pi^e \frac{1 - \ln u}{u^2} du \quad &? \quad 0 && x = e, y = \pi \\
-\int_e^\pi \frac{1 - \ln u}{u^2} du \quad &? \quad 0 \\
\int_e^\pi \frac{\ln u - 1}{u^2} du \quad &? \quad 0 && \text{Integrand} > 0 \text{ within } (e, \pi) \text{ and strictly increasing} \\
\int_e^\pi \frac{\ln u - 1}{u^2} du \quad &> \quad 0 \\
e^\pi &> \pi^e
\end{aligned}
$$

Reference: Montana State University

Chapter 4

Classic Problems

4.1 Coupon Collector's Problem

There are 5 possible coupons. Everyday, you are given a random coupon. How many days, on average, would you need to collect all 5 coupons?

Can you generalize your answer for N coupons?

4.1.1 Solution

The total number of days to collect all 5 coupons is the number of days it takes to collect the first unique coupon plus the number of days it takes to collect the second unique coupon ...

$$X_{tot} = X_1 + X_2 + X_3 + X_4 + X_5$$

These follow a Geometric probability distribution. More specifically $X_1 \sim Geom(1), X_2 \sim Geom(4/5), X_3 \sim Geom(3/5)$... What does this even mean? It means the value for X_i are scattered among probabilities.

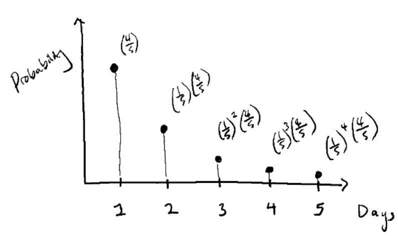


Figure 4.1: Note the "days" here represent the value for X_2 not X_{tot}. As you can see, the value are scattered among probabilities

As seen in the figure above, X_2 is scattered among probabilities. For X_2 to be 1 the probability is 4/5. For X_2 to be 2 the probability is $\frac{1}{5}\frac{4}{5}$. More formally, $P(X_i = d) = (1-p)^{d-1}p$ where d is the number of days.

What is essential to know for this problem is that for geometric distributions, the expected value is the reciprocal of the probability that defined the distribution. The proof can be found in references (cs70 notes).

$$X \sim Geom(p) \Rightarrow E[X] = 1/p$$

Using the fact that they all follow Geometric Distributions and the rule above for the expectation of Geometric Distributions, we can solve our problem.

$E[X_{tot}] = E[X_1] + E[X_2] + E[X_3] + E[X_4] + E[X_5]$ Rules for expected value

$E[X_{tot}] = 1 + \frac{5}{4} + \frac{5}{3} + \frac{5}{2} + 5$ Using $X \sim Geom(p) \Rightarrow E[X] = 1/p$ rule

$E[X_{tot}] = 5(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5})$ Simplify

$E[X_{tot}] \approx 11.41$ days

More generally:

$$E[X_{total}] = N \cdot (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{N}) = N \cdot \sum_{n=1}^{N} \frac{1}{n}$$

References

- https://funnyscar.com/writings/coupon-collectors
- CS70 https://www.eecs70.org/assets/pdf/notes/n19.pdf

4.2 Brachistochrone Problem

Given two points, A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time. (exclude friction)

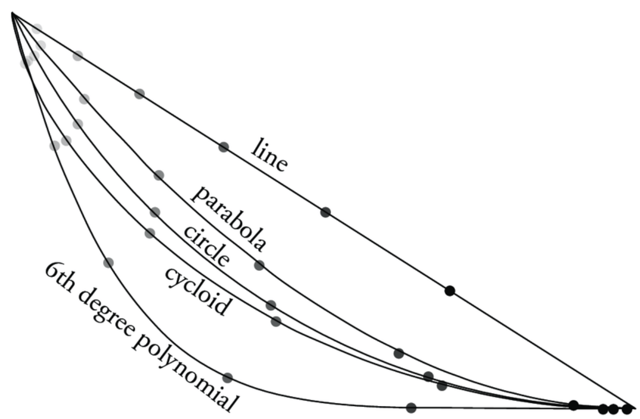


Figure 4.2: What is the fastest path for a frictionless ball?

4.2.1 Solution

There are actually many many solutions to this over the years. Johann Bernoulli posed this problem to the great mathematicians of the time. I'm going to use Bernoulli's argument, which is perhaps the simplest.

4.2.2 Bernoulli's Argument

By *Fermat's Principle*, light follows the path of least time between two points. Bernoulli used this idea to find the path of light in a medium such that the speed increases following a constant acceleration. Similarly, in our original problem, our ball is gaining speed under constant gravitational acceleration, and we want the path of least time.

Imagine we have a substance such that the refractive index uniformly gets smaller as we increase depth. Mimics the constant acceleration of the ball.

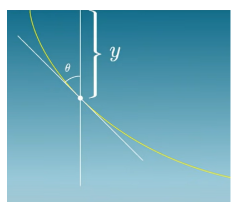
We can start with Snell's Law. $n_i sin\theta_i = n_f sin\theta_f$

$$n_i sin\theta_i = n_1 sin\theta_1 = ... = n_f sin\theta_f \qquad \text{Snell's Law for all the layers}$$

For an arbitrary slice, the $nsin\theta$ term as a whole will be equal to the last slice's term.

$$\frac{c}{v} sin\theta = \frac{csin(\pi/2)}{v_m} = \frac{c}{v_m} \qquad \text{last slice where } \theta = \pi/2 \text{ and speed is highest}$$

$$\frac{sin\theta}{v} = \frac{1}{v_m} \qquad \text{where } v_m \text{ is a constant}$$



$$\frac{sin\theta}{v} = \frac{1}{v}\frac{dx}{ds} \qquad \text{using the relationship } sin\theta = \frac{dx}{ds}$$

$$\frac{1}{v}\frac{dx}{ds} = \frac{1}{v_m} \qquad \text{Relationships defined above}$$

$$(v_m dx)^2 = (vds)^2 \qquad \text{Rearranging}$$

$$v_m^2 dx^2 = v^2(dx^2 + dy^2) \qquad \text{because } ds^2 = dx^2 + dy^2$$

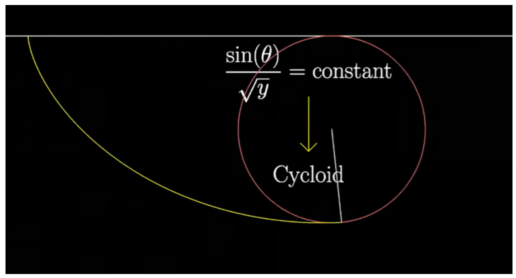
$$dx = \frac{vdy}{\sqrt{v_m^2 - v^2}} \qquad \text{Rearranging}$$

Due to conservation of energy: $\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$. Since no energy is lost from friction and potential energy is directly translated to kinetic energy, velocity directly relates to the height traveled. The path will soon bend until it is completely horizontal, where it would have reached a height of D.

$$dx = \frac{\sqrt{2gy}}{\sqrt{2gD - 2gy}}dy = \sqrt{\frac{y}{D-y}}dy \quad v_m = \sqrt{2gD}, v = \sqrt{2gy}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{D-y}{y}$$

This is the differential equation of a cycloid where $2r = D$



References

- 3b1b
- Wikipedia page

Chapter 5

Bad

Too complicated, bad bulky solution, more pain than pleasure.